The Vectorially Matroidal Structure of Generalized Rough Sets Based on Relations

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Abstract—This paper establishes the vectorially matroidal structure of generalized rough sets based on relations. Any relation induces the vectorial matroid through the representing matrix of the relation. And in the neighborhood of generalized rough sets based on relations is connected with the circuit of the vectorial matroid. On the other hand, the equivalence relation is induced by a relation through the circuit. And the relationship between two inductions is studied. Results show that these two inductions are converse if and only if the relation is an equivalence one.

Keywords—Rough set; vectorial matroid; circuit; relation; neighborhood; representing matrix.

I. INTRODUCTION

Rough set theory has achieved rapid development in the last decade. It has a variety of applications in many areas, such as in knowledge reduction [1], [2] and feature selection [3], [4].

Rough set theory has many significant generalizations. It has been extended to rough sets based on similar relations [5], on tolerance relations [6], on relations [7], covering-based rough sets [8], [9], [10] and probabilistic rough sets [11], [12].

In order to get wider applications, rough set theory has been connected with other theories, such as with matroid theory [13], [14], with topology [15], [16], and with fuzzy sets [17], [18].

This paper bridges generalized rough sets based on relations with matroid theory. Through the representing matrix of a relation, the vectorially matroidal structure is established. On the other hand, the equivalence relation is induced by a matroid through the circuit. And the relationship between two inductions is studied. Results show that these two inductions are converse if and only if the relation is an equivalence one.

The rest of this paper is organized as follows. Section II reviews definitions of matroids and relations. In section III, the vectorial matroid is induced by a relation. Section IV provides an approach to induce the equivalence relation by a matroid. In section V, the two inductions are studied. Section VI concludes this paper.

II. BASIC DEFINITIONS

In this section, we review some definitions of relations and matroids. Firstly, we introduce the binary relation which is fundamental for generalized rough sets.

Definition 1: (Binary relation [19]) Let $U$ be a set, and $U \times U$ the product set of $U$ and $U$. Any subset $R$ of $U \times U$ is called a binary relation on $U$. For all $(x,y) \in U \times U$, if $(x,y) \in R$, we say $x$ has relation $R$ with $y$, and denote this relationship as $xRy$.

Throughout this paper, a binary relation is simply called a relation. In the following definition, three important properties of a relation, namely, the reflexive, symmetric and transitive, are presented.

Definition 2: (Reflexive, symmetric, and transitive [19]) Let $R$ be a relation on $U$.

If for all $x \in U$, $xRx$, we say $R$ is reflexive.

If for all $x, y \in U$, $xRy$ implies $yRx$, we say $R$ is symmetric.

If for all $x, y, z \in U$, $xRy$ and $yRz$ imply $xRz$, we say $R$ is transitive.

Through the above three important properties of a relation, the equivalence relation is introduced.

Definition 3: (Equivalence relation [19]) Let $R$ be a relation on $U$. If $R$ is reflexive, symmetric and transitive, we say $R$ is an equivalence relation on $U$.

This paper establishes the vectorially matroidal structure of generalized rough sets based on relations. And in the following two definitions, matroids and circuits are introduced.

Definition 4: (Matroid [20]) A matroid is a pair $M = (U, I)$ consisting a finite set $U$ and a collection $I$ of subsets of $U$ satisfying the following three conditions:

1) $\emptyset \in I$;
2) If $A \in I$, and $B \subseteq A$, then $B \in I$;
3) If $A, B \in I$, and $|A| < |B|$, then there exists $c \in B - A$ such that $A \cup \{c\} \in I$, where $|A|$ denotes the cardinality of $A$.

Let $M = (U, I)$ be a matroid. Each subset in $I$ is called an independent set of $M$. A set is called a dependent set of a matroid if the set is not an independent set. Based on the dependent sets, we introduce the circuits of a matroid.

Definition 5: (Circuit [20]) Let $M = (U, I)$ be a matroid. A minimal dependent set of $M$ is called a circuit. And the set of all circuits of $M$ is denoted by $C(M)$. 

III. THE VECTORIAL MATROID INDUCED BY A RELATION

In this section, we provide an approach to induce a matroid by a relation. Through the representing matrix of a relation, linearly independent sets are constructed in generalized rough sets based on relations. Based on them, we establish the vectorially matroidal structure. Firstly, the representing matrix of a relation is presented.

Definition 6: (Representing matrix [21]) Let \( R \) be a relation on \( U = \{u_1, \ldots, u_n\} \). \( A_R = (a_{ij})_{n \times n} \) is called the representing matrix of \( R \) where

\[
a_{ij} = \begin{cases} 
1, & (u_i, u_j) \in R, \\
0, & (u_i, u_j) \notin R.
\end{cases}
\]

And we suppose \( A_R = [a_1, \ldots, a_n]^T \) is a \( n \) dimensional column vector.

Based on the representing matrix, we connect generalized rough sets based on relations with linear algebra. In the following definition, linearly independent sets of a relation are defined to induce the vectorial matroid by the relation.

Definition 7: (Linearly independent) Let \( R \) be a relation on \( U \). We define \( I(R) = \{X = \{u_{j1}, \ldots, u_{jm}\} \subseteq U \mid a_{j1}, \ldots, a_{jm} \text{ are linearly independent}\} \). We call \( I(R) \) the collection of linearly independent sets of \( R \).

The following proposition shows that the vectorial matroid is induced by a relation through linearly independent sets. Therefore, it lays a sound foundation for studying generalized rough sets based on relations in the matroidal structure.

Proposition 1: Let \( R \) be a relation on \( U \). Then \( M(R) = (U, I(R)) \) is a matroid. We call it the vectorial matroid induced by \( R \).

In order to illustrate the vectorial matroid induced by a relation, we give an example.

Example 1: Let \( U = \{u_1, u_2, u_3, u_4\} \) and \( R = \{(u_1, u_1), (u_2, u_2), (u_3, u_3), (u_4, u_1), (u_4, u_3)\} \). Then the representing matrix of \( R \) is \( A_R = \begin{pmatrix} 
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix} \).

The column vectors corresponding to \( u \) column vector corresponding to \( U \), \( a \), \( b \), \( c \), \( d \), \( e \), \( f \), \( g \), \( h \), \( i \), \( j \), \( k \), \( l \), \( m \), \( n \), \( o \), \( p \), \( q \), \( r \), \( s \), \( t \), \( u \), \( v \), \( w \), \( x \), \( y \), \( z \).

In the following two propositions, the relationship between the neighborhood and the vectorial matroid induced by a relation is established. In fact, for any element in a universe, its neighborhood is nonempty if and only if it is an independent set of the vectorial matroid.

Proposition 2: Let \( R \) be a relation on \( U \), and \( M(R) = (U, I(R)) \) the vectorial matroid induced by \( R \). For any \( u \in U \), \( RN(u) \neq \emptyset \iff \{u\} \in I(R) \).

Proof: For any \( u \in U \), \( RN(u) \neq \emptyset \iff \{u\} \in I(R) \).

Proposition 3: Let \( R \) be a relation on \( U \), and \( M(R) = (U, I(R)) \) the vectorial matroid induced by \( R \). For all \( u, v \in U \), \( RN(u) \neq RN(v) \iff \{u, v\} \in I(R) \).

Proof: For all \( u, v \in U \), \( RN(u) \neq RN(v) \iff \{u, v\} \in I(R) \).

Proposition 3 provides a sufficient and necessary condition for a subset with only two elements and the vectorial matroid. In the following proposition, we study a more general situation, that is, an arbitrary subset and the vectorial matroid. However, the following statement do not hold generally.

Let \( R \) be a relation on \( U \), and \( M(R) = (U, I(R)) \) the vectorial matroid induced by \( R \). For any \( X \subseteq U \), \( X \in I(R) \) if and only if \( RN(x) \neq RN(y) \) for all \( x, y \in X \) and \( x \neq y \). A counterexample is given as follows.

Example 2: Let \( U = \{a, b, c\} \) and \( R = \{(a, a), (b, b), (c, a), (c, b)\} \). Then the vectorial matroid induced by \( R \) is \( M(R) = (U, I(R)) \), where \( I(R) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\} \). Suppose \( X = U \), then for all \( x, y \in X \) and \( x \neq y \), \( RN(x) \neq RN(y) \) since \( RN(a) = \{a\} \), \( RN(b) = \{b\} \) and \( RN(c) = \{a, b\} \). But \( X \notin I(R) \).

Interestingly, it is true when the relation is an equivalence relation.

Proposition 4: Let \( R \) be an equivalence relation on \( U \), and \( M(R) = (U, I(R)) \) the vectorial matroid induced by \( R \). For any \( X \subseteq U \), \( X \in I(R) \) iff \( x \neq y \) implies \( RN(x) \neq RN(y) \) for all \( x, y \in X \).

Proof: \((\Rightarrow)\): For any \( X \subseteq U \), if there exist \( x, y \in X \), \( x \neq y \) and \( RN(x) = RN(y) \), then the column vectors corresponding to \( x, y \) are linearly dependent. Thus it is straightforward that the column vectors corresponding to \( X \) are linearly dependent, which is contradictory with \( X \in I(R) \). Hence \( X \in I(R) \) iff \( RN(x) \neq RN(y) \) for all \( x, y \in X \) and \( x \neq y \). 

\((\Leftarrow)\): For any \( X \subseteq U \), and \( RN(x) \neq RN(y) \) for all \( x, y \in X \) and \( x \neq y \), we need to prove \( X \in I(R) \); in other words, the column vectors corresponding to \( X \) are linearly independent. The column vectors corresponding to \( X \) are denoted by \( \{e_i \mid x \in X\} \). Since \( R \) is an equivalence relation, then \( RN(u) = RN(v) \cap RN(u) \neq \emptyset \) for all \( u, v \in U \). Thus for all \( x, y \in X \) and \( x \neq y \), \( RN(x) \neq RN(y) \).
y, \ c_x \land c_y = [0, \ldots, 0]^T \text{ where } c_x \land c_y = [x_1 \land y_1, \ldots, x_n \land y_n]^T = [\min\{x_1, y_1\}, \ldots, \min\{x_n, y_n\}]^T, c_x = [x_1, \ldots, x_n]^T \text{ and } c_y = [y_1, \ldots, y_n]^T. \text{ Therefore, } \sum_{x \in X} k_x c_x = [0, \ldots, 0]^T \iff k_x = 0 \text{ for all } x \in X. \text{ Hence the column vectors corresponding to } X \text{ are linearly independent; in other words, } X \in \mathbb{I}(R).}

Based on Proposition 4, we present the expression of the vectorial matroid induced by an equivalence relation.

**Proposition 5:** Let $R$ be an equivalence relation on $U$, and $M(R) = (U, \mathbb{I}(R))$ the vectorial matroid induced by $R$. Then $\mathbb{I}(R) = \{X \subseteq U \mid x, y \in X, x \neq y \implies (x, y) \notin R\}$.

**IV. THE EQUIVALENCE RELATION INDUCED BY A MATROID**

In this section, the converse question, that is, how a matroid induces a relation, is studied. We provide an approach to induce the equivalence relation by a matroid through the circuit.

**Definition 9:** Let $M = (U, \mathbb{I})$ be a matroid, and $C(M)$ the sets of all circuits. We define the relation $R(M)$ induced by $M$ as follows: for all $u, v \in U$

$u R(M) v \text{ if and only if } \{u, v\} \in C(M)$.

According to Definition 9, it is proved that the relation induced by a matroid is an equivalence one.

**Proposition 6:** Let $M = (U, \mathbb{I})$ be a matroid, and $R(M)$ the relation induced by $M$. Then $R(M)$ is an equivalence relation on $U$.

**V. THE RELATIONSHIP BETWEEN TWO INDUCTIONS**

In section III, the vectorial matroid is induced by a relation. And in section IV, the equivalence relation is induced by a matroid. In this section, we study the relationship between the two inductions. First of all, the connection between the neighborhood and the circuit of the vectorial matroid is established. In fact, a subset having only two elements is a circuit if and only if the neighborhoods of them are the same.

**Proposition 7:** Let $R$ be a relation on $U$ and $M(R)$ the vectorial matroid induced by $R$. Then $RN(u) = RN(v) \iff \{u, v\} \in C(M(R))$.

Based on Proposition 7, we establish the relationship between a relation and the relation induced by the vectorial matroid.

**Proposition 8:** Let $R$ be a relation on $U$ and $M(R)$ the vectorial matroid induced by $R$. If $R$ is reflexive, then $R(M(R)) \subseteq R$.

**Proof:** For all $(u, v) \in R(M(R))$, if $u = v$, then $(u, v) \in R$ since $R$ is reflexive. If $u \neq v$, then $\{u, v\} \in C(M(R))$. Thus $RN(u) = RN(v)$; in other words, $v \in RN(u)$. Hence $(u, v) \in R$. This completes the proof.

The following proposition shows that two inductions are converse when the relation is an equivalence one.

**Proposition 9:** Let $R$ be a relation on $U$ and $M(R)$ the vectorial matroid induced by $R$. Then $R$ is an equivalence relation iff $R(M(R)) = R$.

**Proof:** ($\Leftarrow$:): According to Proposition 6, it is straightforward.

($\Rightarrow$:): According to Proposition 8, $R(M(R)) \subseteq R$ is straightforward. So we only need to prove $R \subseteq R(M(R))$. In fact, when $R$ is an equivalence relation, for all $(u, v) \in R$, $RN(u) = RN(v)$. Then $\{u, v\} \in C(M(R))$. Hence $(u, v) \in R(M(R))$.

**VI. CONCLUSIONS**

This paper establishes the vectorially matroidal structure of generalized rough sets based on relations. Through the representing matrix, any relation induces the vectorial matroid. Conversely, the equivalence relation is induced by a matroid. And the relationship between two inductions is studied. Specifically, the two inductions are converse when the relation is an equivalence one. We will further study the properties and applications of generalized rough sets based on relations in the vectorially matroidal structure.

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